AD-A128 027 SIGNIFICANCE POINTS FOR SOME TESTS OF UNIFORMITY ON THE SPHERE REVISION(U) ROCHESTER UNIV NY GRADUATE SCHOOL OF MANAGEMENT J KEILSON ET AL. NOV 82 AFOSR-TR-83-0329 1/1

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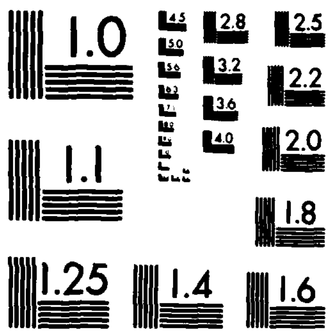

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REPORT DOCUMENTATION PAGE

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1. REPORT NUMBER	AFOSR-TR- 83-0329
2. GOVT ACCESSION NO.	AD-A128027
3. RECIPIENT'S CATALOG NUMBER	
4. TITLE (and Subtitle)	SIGNIFICANCE POINTS FOR SOME TESTS OF UNIFORMITY ON THE SPHERE
5. TYPE OF REPORT & PERIOD COVERED	TECHNICAL
6. PERFORMING ORG. REPORT NUMBER	Working Paper Series No. 8029
7. AUTHOR(s)	J. Keilson, D. Petrondas, U. Sumita and J. Wellner
8. CONTRACT OR GRANT NUMBER(s)	AFOSR-79-0043
9. PERFORMING ORGANIZATION NAME AND ADDRESS	The Graduate School of Management University of Rochester Rochester NY 14627
10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS	PE61102F; 2304/A5
11. CONTROLLING OFFICE NAME AND ADDRESS	Mathematical & Information Sciences Directorate Air Force Office of Scientific Research Bolling AFB DC 20332
12. REPORT DATE	JUL 80; Revised NOV 82
13. NUMBER OF PAGES	31
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)	
15. SECURITY CLASS. (of this report)	UNCLASSSIFIED
15a. DECLASSIFICATION/DOWNGRADING SCHEDULE	

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

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ITEM #20, CONTINUED:

Beran and Gine for n = 2;

(2) presenting some percentage points for selected small and moderate sample sizes obtained by Monte-Carlo methods;

(3) evaluating numerically the cumulative distribution functions and significance points of the limiting distributions via the Laguerre transform method (Keilson and Nunn (1979), Keilson, Nunn and Sumita (1981), and Sumita (1981)).

AFOSR-TR- 83-0329

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Working Paper Series No. 8029

July, 1980

Revised November 1982

The Graduate School of Management
University of Rochester

Key Words and Phrases: Small samples, Gine's statistics, Weighted sums of independent Chi-square variates, convolution, the Laguerre transform method

This work was supported in part by the United States Air Force, Office of Scientific Research under grant No. AFOSR-79-0043.

83 05 06 - 184

Significance points for some tests of uniformity on the sphere

J. Keilson, D. Petrondas, U. Sumita, J. Wellner

Abstract

Beran (1968) and Giné (1975) have proposed several omnibus tests for uniformity on the unit sphere in three dimensional Euclidean space. While several authors have contributed to providing approximate percentage points for the limiting distributions, no tables of the limiting distributions, percentage points thereof, or finite sample distributions or percentage points have been available. In this paper The authors ~~we~~ fill this gap by:

(1) finding the exact distributions of the statistics of Beran and Giné for n = 2;

(2) presenting some percentage points for selected small and moderate sample sizes obtained by Monte-Carlo methods;

(3) evaluating numerically the cumulative distribution functions and significance points of the limiting distributions via the Laguerre transform method, (Keilson and Nunn (1979), Keilson, Nunn and Sumita (1981), and Sumita (1981)).

§1. Introduction

Beran (1968) and Giné (1975) have proposed several omnibus tests for uniformity on the unit sphere $S = \{\underset{\sim}{x} \in R^3: |\underset{\sim}{x}| = 1\}$ in three-dimensional Euclidean space. These tests are consistent against all alternatives and are locally most powerful for specific alternatives. Beran and Giné have shown that the limiting distributions of these statistics, under the null hypothesis of uniformity, are those of weighted sums of independent Chi-square variables. While Prentice (1975) has applied the methods of Zolotarev (1961) and Hoeffding (1964) to provide approximate percentage points for the limiting distributions, no tables of the limiting distributions, percentage points thereof, or finite sample distributions or percentage points have been available.

Our purpose here is to fill this gap by:

(1) finding the exact distributions of the statistics of Beran and Giné for n = 2;

(2) presenting some percentage points for selected small and moderate sample sizes obtained by Monte-Carlo methods;

(3) using the Laguerre transform method - Keilson and Nunn (1979), Keilson, Nunn and Sumita (1981), and Sumita (1981) to compute the cumulative distribution functions and significance points of the limiting distributions.

In Section 2, we summarize the result on the limiting distributions due to Beran and Giné and give the exact distributions for n = 2. Section 3 contains the Monte-Carlo results for finite sample sizes and description of the methods used. We discuss, in Section 4, the numerical procedure for

evaluating the limiting distributions via the Laguerre transform method. An application is given in Section 5 where we test uniformity of orientation of dendritic fields in the retinas of cats subject to controlled visual environments. The numerical results are summarized in Section 6 in tables and graphs.

§2. The statistics: limiting distributions and exact distributions for $n = 2$

Let $\underline{X}_1,\ldots,\underline{X}_n$ be independent and identically distributed unit vectors in R^3 with distribution ν (so $\nu(S) = 1$, where $S = \{\underline{x} \in R^3: |\underline{x}| = 1\}$). Let $\theta_{ij} \equiv \underline{X}_i \hat{} \underline{X}_j \equiv \arccos(\underline{X}_i \cdot \underline{X}_j) =$ the angle between $\underline{X}_i$ and $\underline{X}_j$ for $i,j = 1,2,\ldots n$.

For testing the null hypothesis that ν is the uniform distribution on S, Beran (1968) and Giné (1975) have suggested the statistics

$$Y_n^{odd} \equiv n^{-1} \sum_{i=1}^{n} \sum_{j=1}^{n} \{1 - (2/\pi)\theta_{ij}\} \tag{2.1}$$

$$Y_n^{even} \equiv n^{-1} \sum_{i=1}^{n} \sum_{j=1}^{n} \{\tfrac{1}{2} - (2/\pi)\sin\theta_{ij}\} \tag{2.2}$$

$$Y_n \equiv Y_n^{odd} + Y_n^{even} \quad . \tag{2.3}$$

Giné proposes Y_n^{even} and finds that Y_n^{odd} is simply Beran's form of Ajne's statistic: $Y_n^{odd} = 4T_n$ (Beran, 1968). It is also known that $Y_n^{odd} = 4A_{2,n}$ (Prentice, 1978) and $Y_n^{even} = G_{2,n}$ (Prentice).

Let $\{Z_j\}_{j=1}^{\infty}$ be independent Chi-square random variables with $2j+1$ degrees of freedom (so $Z_{2k-1} \sim \chi^2_{4k-1}$, $Z_{2k} \sim \chi^2_{4k+1}$). Set

$$a_{2k-1}^2 = (2k-1)^{-2}[(\tfrac{1}{2})_k/k!]^2 \tag{2.4}$$

$$a_{2k}^2 = (2k-1)^{-1}(2k+2)^{-1}[(\tfrac{1}{2})_k/k!]^2 \tag{2.5}$$

where $(\frac{1}{2})_k = \frac{1}{2}\cdot(\frac{1}{2}+1)\ldots\ldots(\frac{1}{2}+k-1)$ and define

$$Y^{odd} = \sum_{k=1}^{\infty} a_{2k-1}^2 Z_{2k-1} \,, \tag{2.6}$$

$$Y^{even} = \sum_{k=1}^{\infty} a_{2k}^2 Z_{2k} \,, \tag{2.7}$$

$$Y = Y^{odd} + Y^{even} \quad . \tag{2.8}$$

Theorem 1 (Giné): If ν is the uniform distribution on S, then

$$\lim_{n\to\infty} P[Y_n^{odd} \le y] = P[Y^{odd} \le y] \quad ,$$

$$\lim_{n\to\infty} P[Y_n^{even} \le y] = P[Y^{even} \le y] \quad ,$$

and $\lim_{n\to\infty} P[Y_n \le y] = P[Y \le y]$ for all $y \in R$.

Prentice (1978) uses the methods of Zolotarev (1961) and Hoeffding (1964) to provide approximate percentage points for these limiting distributions. We will compute the distributions with precision via the Laguerre transform method [Keilson and Nunn (1979), Keilson, Nunn and Sumita (1981), and Sumita (1981)] in Section 4 and present tables and graphs in Section 6.

We now consider the distributions of Y_n^{odd}, Y_n^{even} and Y_n for n = 2.

Theorem 2: If ν is the uniform distribution on S, then

$$P[Y_2^{odd} \le y] = \tfrac{1}{2}\{1 + \cos[\tfrac{\pi}{2}(y-2)]\} \ , \quad 0 \le y \le 2 \ ,$$

$$P[Y_2^{even} \le y] = \cos\{\text{arc}\sin[\tfrac{\pi}{2}(1-y)]\} \ , \ 1 - \tfrac{2}{\pi} \le y \le 1 \ ,$$

$$P[Y_2 \le y] = \tfrac{1}{2}\{1 + \cos[g^{-1}(\tfrac{\pi}{2}(3-y))]\} \ , \ 0 \le y \le 3 \quad ,$$

where $g(t) \equiv t + \sin t$, $0 \le t \le \pi$, and g^{-1} denotes the inverse of g.

Proof: First note that under uniformity $P[\theta_{12} \le y] = (4\pi)^{-1} \int_0^{2\pi}\int_0^{y} \sin\theta d\theta d\phi = \frac{1}{2}(1 - \cos y)$, $0 \le y \le \pi$. Then, writing $Y_2^{odd} = \frac{1}{2}\{1 + 1 + (1 - (2/\pi)\theta_{12}) + (1 - \frac{2}{\pi}\theta_{21})\} = 2 - (\frac{2}{\pi})\theta_{12}$, $Y_2^{even} = \frac{1}{2}\{\frac{1}{2} + \frac{1}{2} + (\frac{1}{2} - \frac{2}{\pi}\sin\theta_{12}) + (\frac{1}{2} - \frac{2}{\pi}\sin\theta_{21}) = 1 - (\frac{2}{\pi})\sin\theta_{12}$, and $Y_2 = 3 - (\frac{2}{\pi})(\theta_{12} + \sin\theta_{12})$, the stated distributions are easily obtained by straightforward computation.$\square$

For future reference, we record some moments in the following Proposition:

Proposition 3. Under uniformity we have

$$E(Y_n^{odd}) = E(Y^{odd}) = 1$$

$$E(Y_n^{even}) = E(Y^{even}) = \frac{1}{2}$$

$$E(Y_n) = E(Y) = \frac{3}{2}$$

while

$$Var(Y_n^{odd}) = (1 - \frac{1}{n})(2 - \frac{16}{\pi^2}) \xrightarrow{n\to\infty} (2 - \frac{16}{\pi^2}) = .37886 \ldots = Var[Y^{odd}] ,$$

$$Var(Y_n^{even}) = (1 - \frac{1}{n})(\frac{16}{3\pi^2} - \frac{1}{2}) \xrightarrow{n\to\infty} (\frac{16}{3\pi^2} - \frac{1}{2}) = .04038\ldots = Var[Y^{even}] ,$$

and

$$Var(Y_n) = (1 - \frac{1}{n})(\frac{3}{2} - \frac{32}{3\pi^2}) \xrightarrow{n\to\infty} (\frac{3}{2} - \frac{32}{3\pi^2}) = .41924\ldots = Var[Y] .$$

Furthermore, the third and fourth cumulants of Y^{odd} and Y^{even} are given by

$$K_3(Y^{odd}) = .375219\ldots , \quad K_4(Y^{odd}) = .56252\ldots ,$$

$$K_3(Y^{even}) = .0098016\ldots , \quad K_4(Y^{even}) = .00366374\ldots .$$

Proof:

The finite sample means and variances are easily obtained by elementary methods upon noting that θ_{ij} and $\theta_{ij'}$ are independent for $j \neq j'$; the asymptotic means and variances follow immediately by letting $n \to \infty$. The cumulants of Y^{odd} and Y^{even} are easily computed using the following easily

derived formula for the m^{th} cumulant K_m of $\sum_{j=1}^{\infty} \beta_j \chi^2_{f_j}$:

$$K_m = 2^{m-1}(m-1)! \sum_{j=1}^{\infty} f_j \beta_j^m \quad . \qquad \square$$

§3. Monte-Carlo simulations

In our simulations, a function subprogram called RAND (University of Rochester file #311·7·500, Computer Center) and the IBM/360 computer are used to generate uniformly distributed random numbers. The method of Marsaglia (1972) is then employed to generate points from a distribution on the unit 3-sphere. Using n such random points, the statistics Y_n^{odd}, Y_n^{even} and Y_n defined in Section 2 are computed. With Monte-Carlo samples of size 5000 for n = 100, and 20000 for n = 5, 10, 20 and 40, the percent points of the above statistics are estimated for the significance levels α = .20, .10, .05, .025, .01 and .001. Those empirical finite sample percentage points are further smoothed in the following manner. For each level α and sample size n, let a smooth function $\hat{Y}_{\alpha,n}$ be defined by

$$\hat{Y}_{\alpha,n} = a_{\alpha,\infty} + \frac{4}{n^2}(a_{\alpha,2} - a_{\alpha,\infty}) + b_{\alpha,n}\left(\frac{1}{n} - \frac{2}{n^2}\right) . \tag{3.1}$$

Here $\hat{Y}_{\alpha,n}$ is the estimated Y value for each n at level α. $a_{\alpha,2}$ and $a_{\alpha,\infty}$ are the Y values at level α for n = 2 and n = ∞, respectively. $a_{\alpha,2}$ is found from Theorem 2, and $a_{\alpha,\infty}$ from the Laguerre transform method to be described in Section 4. $b_{\alpha,n}$ is the estimated slope obtained from the original Monte-Carlo results via the straightforward linear regression. It should be noted that the smooth function (3.1) coincides with the known values of Y when n = 2 or n = ∞.

The estimated smoothed percentage points of the three statistics are presented in Table 6.1 of Section 6.

§4. Evaluation of the limiting distributions via the Laguerre transform

We have seen in Section 2 that Beran's and Giné's limiting statistics under the null hypothesis of uniformity are infinite sums of independent and scaled Chi-square variates, i.e., Y^{odd}, Y^{even} and Y as given in (2.6), (2.7) and (2.8). In this section, we discuss numerical evaluation of Y^{odd}, Y^{even} and Y and propose a Laguerre transform approach to be described.

It is natural to decompose the infinite sum $Y = \sum_{j=1}^{\infty} a_j^2 Z_j$ in (2.8) into two parts, the sum of the first N variates and the remainder, i.e.,

$$Y = S_N + V_N \ ; \quad S_N = \sum_{j=1}^{N} a_j^2 Z_j \ ; \quad V_N = \sum_{j=N+1}^{\infty} a_j^2 Z_j \quad . \tag{4.1}$$

Correspondingly, let

$$Y^{odd} = S_N^{odd} + V_N^{odd} \ ; \quad Y^{even} = S_N^{even} + V_N^{even} \tag{4.2}$$

where S_N^{odd} and V_N^{odd} are the sums of odd index terms of S_N and V_N, respectively. The variates S_N^{even} and V_N^{even} are defined similarly for even index terms. S_N is the finite sum of independent scaled Chi-square variates and such linear combinations may be regarded as positive-definite quadratic forms in normal variables. Many papers have been published on the distribution of such quadratic forms, and the reader is referred to Johnson and Kotz (1970, Ch. 29) for a comprehensive survey of the literature. An excellent approach to the numerical evaluation of such distributions is that of Johnson, Kotz and Boyd (1967). They expand the distribution function in a series of generalized Laguerre functions and evaluate it efficiently by taking advantage of the recurrence relation of the Laguerre functions. When this procedure is applied directly to the distribution of

S_N, however, one encounters numerical difficulty. The coefficients a_j^2 decreases rapidly and the distributions of $a_j^2 Z_j$ become very concentrated, resulting in quite slow convergence of the corresponding Laguerre series. Our procedure restructures the method of Johnson, Kotz and Boyd and overcomes this numerical difficulty.

The Laguerre transform method for convolving functions has been introduced by Keilson and Nunn (1979), Keilson, Nunn and Sumita (1981), and further studied by Sumita (1981). The method has advantages of accuracy and speed which make it an attractive candidate for problems of this type. The Laguerre transform method has peculiarities and limitations, however, which require careful refinement for particular contexts, such as that here. The reader is referred to three basic papers for the underlying theory. A brief summary is given in Appendix A for the convenience of the reader. Our basic strategy is to approximate Y^{odd} and Y^{even} by

$$Y_N^{odd*} = S_N^{odd} + V_N^{odd*} \quad ; \quad Y_N^{even*} = S_N^{even} + V_N^{even*} \tag{4.3}$$

where V_N^{odd*} and V_N^{even*} are the Gamma variates having the same first two moments of V_N^{odd} and V_N^{even}, respectively. Correspondingly, Y is approximated by

$$Y_N^* = Y_N^{odd*} + Y_N^{even*} = S_N + V_N^* \quad . \tag{4.4}$$

In the subsections to follow, the Laguerre transform procedure is described for evaluating the distribution of Y_N^{odd*}, Y_N^{even*} and Y_N^*. The

validity of the results is also examined.

(A) <u>The Laguerre sharp coefficients of the Gamma variate $\Gamma(\alpha, 2\beta)$</u>

Let $\Gamma(\alpha, 2\beta)$, $\alpha,\beta > 0$, be the Gamma variate with p.d.f.

$$g(x) = \frac{1}{\Gamma(\alpha)(2\beta)^{\alpha}} x^{\alpha-1} e^{-\frac{1}{2\beta}x}, \quad 0 \le x < \infty \quad . \tag{4.5}$$

It is clear that the variates $a_j^2 z_j$ belong to this family with $\alpha = j + \frac{1}{2}$ and $\beta = a_j^2$. Hence the Laguerre sharp coefficients $(g_n^{\#})_0^{\infty}$ of $g(x)$ provides a basic tool for the procedure. From the Laplace transform $\gamma(s) = \int_0^{\infty} e^{-sx} g(x)dx = (1 + 2\beta s)^{-\alpha}$ and the identity $T_g^{\#}(u) = \sum_{n=0}^{\infty} g_n^{\#} u^n = \gamma(\frac{1}{2} \cdot \frac{1+u}{1-u})$, those coefficients are found by (cf. Sumita (1981), Section 6.2)

$$g_n^{\#} = (1+\beta)^{-\alpha} \sum_{m=0}^{n} b_{n-m} c_m \tag{4.6}$$

where

$$\begin{aligned} b_n &= \prod_{r=1}^{n} \left(1 - \frac{1+\alpha}{r}\right), \quad n \ge 1,\ b_0 = 1 \\ c_n &= \left(\frac{1-\beta}{1+\beta}\right)^n \prod_{r=1}^{n} \left(1 - \frac{1-\alpha}{r}\right), \quad n \ge 1,\ c_0 = 1 \quad . \end{aligned} \tag{4.7}$$

The accuracy and efficiency of the Laguerre transform method depend heavily on one's ability to represent the functions present with a sequence of Laguerre coefficients of reasonable length (say, at most around 500 coefficients to attain 5 digits accuracy). As studied theoretically in Keilson, Nunn and Sumita (1981) and Sumita (1981), the Laguerre transform method, when applied in a straightforward manner, cannot tolerate functions

too closely concentrated at zero or functions too great in extent. For the Gamma variate, this point can be observed explicitly in (4.6) and (4.7). When α is extremely large, b_n and c_n become so large in absolute value that the computer may not tolerate them. The other numerical difficulty arises when β is extremely small or large. In this case, the ratio $\left|\frac{1-\beta}{1+\beta}\right|$ becomes very close to 1, and one would expect (c_n) to have a long tail. This, in turn, implies a long tail of $g_n^{\#}$. Fortunately, these numerical difficulties can be avoided by taking advantage of the divisibility of the Gamma variates and employing scaling. In brief, the first difficulty can be solved through the identity $\gamma(s) = [\gamma_M(s)]^M$, where $M > 0$ and $\gamma_M(s) = (1 + 2\beta s)^{-\alpha/M}$ corresponding to the Gamma variate $\Gamma(\alpha/M, 2\beta)$. By an appropriate choice of positive integer M, the Laguerre sharp coefficients of $\Gamma(\alpha/M, 2\beta)$ are obtained with reasonable length. We then convolve them M times on the lattice to recover the Laguerre sharp coefficients of the original Gamma variate $\Gamma(\alpha, 2\beta)$. For the second numerical difficulty, we replace β by $c\beta$ so that the ratio $\left|\frac{1-c\beta}{1+c\beta}\right|$ becomes well below 1. Then (c_n) in (4.7) decreases rapidly in absolute value and therefore $g_n^{\#}$ decreases rapidly. After the inversion of the Laguerre sharp representation, the proper scale factor for the probability density function is restored.

(B) <u>Algorithms for finding the sharp coefficients of Y^{odd*}, Y^{even*} and Y^*</u>

(1) For a desired small variance of V_N, select N for S_N and V_N. Then choose the dividing factor M for V_N, and the scale factor c in keeping with the conditions of (A). The number L of Laguerre coefficients before truncation will be discussed in (C) below.

(2) Using (4.6) and (4.7) with $\alpha = j + \frac{1}{2}$ and $\beta = ca_j^2$, calculate $(g(j)_n^{\#})_0^L$ of $ca_j^2 Z_j$ for $1 \le j \le N$.

(3) Obtain $(g_{N,n}^{odd\#})_0^L$ of cS_n^{odd} by convolving $(g(j)_n^{\#})_0^L$ for j odd, $1 \le j \le N$. Obtain $(g_{N,n}^{even\#})_0^L$ of cS_N^{even} similarly.

(4) Calculate the means μ_{VN}^{odd} and μ_{VN}^{even} and the variances σ_{VN}^{odd2} and σ_{VN}^{even2} of V_N^{odd} and V_N^{even}, respectively, from (2.6), (2.7) and Proposition 3. Using (4.6) and (4.7) with $\alpha = (\mu_{VN}^{odd}/\sigma_{VN}^{odd})^2/M$ and $\beta = c \cdot \sigma_{VN}^{odd2}/2\mu_{VN}^{odd}$, calculate $(h^{odd\#}(M,N)_n)_0^L$. By convolving $(h^{odd\#}(M,N)_n)_0^L$ M times with itself, find $(h_{N,n}^{odd\#})_0^L$ of cV_N^{odd*}. Obtain $(h_{N,n}^{even\#})_0^L$ of cV_N^{even*} similarly.

(5) Calculate $(f_{N,n}^{odd\#})_0^L$ of cY_N^{odd*} by convolving $(g_{N,n}^{odd\#})_0^L$ and $(h_{N,n}^{odd\#})_0^L$. Obtain $(f_{N,n}^{even\#})_0^L$ of cY_N^{even*} similarly.

(6) Finally, calculate $(f_{N,n}^{\#})_0^L$ of cY_N^* by convolving $(f_{N,n}^{odd\#})_0^L$ and $(f_{N,n}^{even\#})_0^L$.

<u>Remark</u>

When one has the Laguerre sharp coefficients $(f_n^{\#})_0^{\infty}$ of a p.d.f. f(x) on $(0,\infty)$, the inversion of $(f_n^{\#})_0^{\infty}$ to the values of f(x) and its survival function $\bar{F}(x) = \int_x^{\infty} f(x)dx$ can be done in the following manner (cf. Keilson and Nunn (1980)).

(R1) Calculate $(\ell_n(x))_0^{\infty}$ for $0 \le x < \infty$ by

$$\ell_{n+1}(x) = \frac{1}{n+1} [(2n+1-x)\ell_n(x) - n\ell_{n-1}(x)] \quad , \quad n \ge 1$$

where $\ell_0(x) = e^{-\frac{1}{2}x}$.

(R2) Obtain $f_n^+ = \sum_{m=0}^{n} f_m^{\#}$ and calculate $f(x) = \sum_{m=0}^{\infty} f_n^+ \ell_n(x)$.

(R3) Calculate $\bar{f}_n^+ = -2 \sum_{m=0}^{\infty} (-1)^m f_{n+1+m}^{\#}$ and $\bar{F}(x) = \sum_{n=0}^{\infty} \bar{f}_n^+ \ell_n(x)$.

We note that the Laguerre transform bypasses numerical integration. We also note that the algorithm with (R1), (R2) and (R3) produces $\bar{F}_N^*(x/c)$ where $\bar{F}_N^*(x) = P[Y_N^* > x]$.

(C) Validation of the results

For the calculation of the distributions needed, the scaling factor is taken to be c = 40. The two values N = 2 and N = 4 are used and the dividing factor M = 12 is chosen for V_2^* and M = 25 for V_4^*. The length of the Laguerre sharp coefficients is L = 502, which provides 12 digits accuracy of the p.d.f. of $a_j^2 Z_j$ for $1 \le j \le 4$.

There are two different factors which introduce numerical errors, the truncation of the Laguerre sharp coefficients and the Gamma approximation of V_N. In general, it is quite hard to quantify truncation error of the Laguerre coefficients. (Such error bounding has its counterpart in Fourier series theory, where error bounding is known to be extremely difficult.) Theoretical error bounds are available, so far, only for a certain family of functions (cf. Keilson and Sumita (1981) and Sumita (1981)). Extensive numerical evidence, however, suggests that when one chooses L large enough to attain a given accuracy for the following identities, then the function values are likely to satisfy the same accuracy.

$$f(0+) = -\sum_{n=0}^{\infty} n f_n^\# \tag{4.8}$$

$$\int_0^\infty f(x)dx = \sum_{n=0}^{\infty} (-1)^n f_n^\# \tag{4.9}$$

$$\int_0^\infty x f(x)dx = 4x \sum_{n=0}^{\infty} (-1)^n n f_n^\# \tag{4.10}$$

$$\int_0^\infty x^2 d(x)dx = 16 \times \sum_{n=0}^{\infty} (-1)^n n^2 f_n^\# \quad . \tag{4.11}$$

For $N = 4$, $(f_{4,n}^\#)_0^{502}$ of Y_4^* provides 10 digits accuracy for all equations (4.8) through (4.10) and the truncation error seems to be negligible. Even though we may expect the Gamma approximations for the remainders V_N^{odd}, V_N^{even}, and V_N to introduce little error since $Var[V_N]$ drops rapidly (e.g., $Var[V_4] = 6.6 \times 10^{-4}$), no analytical justification is available, and we are forced to take indirect means for testing the validity of the approximation. To test this validity we note that $Y_N^* \xrightarrow{d} Y$ as $N \to \infty$. The c.d.f.'s of Y_2^* and Y_4^* are calculated and compared. The absolute difference of the two c.d.f.'s is found numerically to be bounded by 1×10^{-5}. For a second accuracy check of the Gamma approximation, the third and the fourth cumulants of Y_4^{odd*} and Y_4^{even*} are calculated using the Laguerre sharp coefficients and compared with exact values. The higher moments of Y_4^{odd*} and Y_4^{even*} can be found from (4.10), (4.11) and (R3) of (B) with the identity $f_n^\# = f_n^\dagger - f_{n-1}^\dagger$, $n \geq 1$, and $f_n^\# = f_0^\dagger$. The cumulants needed are then obtained by

$$\begin{aligned} K_3 &= \mu_3 - 3\mu_1\mu_2 + 2\mu_1^3 \\ K_4 &= \mu_4 - 3\mu_2^2 - 4\mu_1\mu_3 + 12\mu_1^2\mu_2 - 6\mu_1^4 \end{aligned} \tag{4.12}$$

where $\mu_i = E[X^i]$. The corresponding exact values are given in Proposition 3. The computations are shown in Table 4.1. The absolute difference is bounded by 3×10^{-6}.

Finally, we study the asymptotic behavior of the survival function of Y, and compare it with the values computed via the Laguerre transform method. From the Laplace transform of Y, one finds through asymptotic analysis that

$$\overline{F}_Y(x) = P[Y > x] \sim \frac{e^{-2x}}{\sqrt{\pi}} \ [2\lambda_0\sqrt{2x} + (\frac{\lambda_0}{\sqrt{2}} + \lambda_1\sqrt{2})\ \frac{1}{\sqrt{x}}) \tag{4.13}$$

as $x \to +\infty$, where $\lambda_0 \approx 4.95221$ and $\lambda_1 \approx -4.27140$. Details are given in Appendix B. In Figure 4.2, the asymptotic expansion of $\overline{F}_Y(x)$ in (4.13) is plotted with the survival function of Y_4^* derived via the Laguerre transform method. Their absolute difference is found to be bounded by 1×10^{-5} for $x > 3.5$.

A similar approximation for the remainder of an infinite sum of independent variates is employed in Sumita (1979) to evaluate the multiple convolutions of the Logistic variates. There a direct accuracy check is possible and accuracy to seven decimal places is attained.

All calculations were carried out on a DEC10 computer, in a time-sharing mode using APL as the programming language. Relevant formulae are coded in a straightforward manner, with no attempt made to optimize the subroutines for speed and accuracy. In spite of this, the results displayed here were obtained with CPU times in seconds with no evidence of numerical problems.

$K_3\ (Y^{odd})$	0.3752193777	$K_4\ (Y^{odd})$	0.5625201543
$K_3\ (Y_4^{odd*})$	0.3752164153	$K_4\ (Y_4^{odd*})$	0.5625200535
Absolute Difference	0.0000029624	Absolute Difference	0.0000001008
$K_3\ (Y^{even})$	0.0098016768	$K_4\ (Y^{even})$	0.0036637420
$K_3\ (Y_4^{even*})$	0.0098007718	$K_4\ (Y_4^{even*})$	0.0036637195
Absolute Difference	0.0000009050	Absolute Difference	0.0000000225

Table 4.1. The cumulants: Y^{odd} and Y^{even} vs. Y_4^{odd*} and Y_4^{even*}

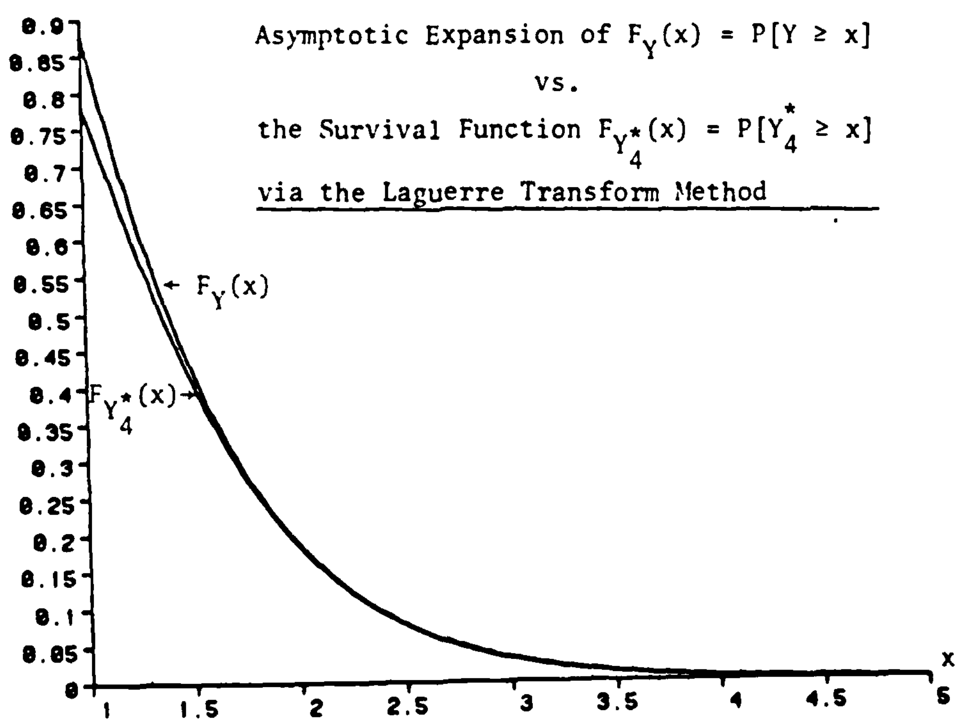

Figure 4.2

§5. Application

In an experiment to determine the effect of different visual stimuli, six cats were divided into three groups of two, and each group was subjected to a different visual stimulus: horizontally polarized light (H), vertically polarized light (V), and unpolarized or "normal" light (N). Orientations of the dendritic fields were then measured at 15 to 16 sites in the retinas of each of the six cats. The complete data set, in coordinates described by Figure 5.1 below, is given in Appendix C.

The orientation of the dendritic fields of the two cats exposed to normal light is, presumably, uniform, with no preferred orientation, and the question is: what is the effect of polarized light? Hence we wish to test the null hypothesis of uniformity of orientation (on the unit hemisphere) of the dendritic fields of the H and V groups. To test this hypothesis, we use Giné's statistic $Y_n^{even} = n^{-1} \sum_{i=1}^{n} \sum_{j=1}^{n} (\frac{1}{2} - (\frac{2}{\pi}) \sin \widehat{X_i X_j})$, where $\widehat{X_i X_j}$ is the angle between the observations X_i and X_j. Since X_i is represented by a vector $\underline{x}_i = [\cos\phi_i \sin\theta_i, \sin\phi_i \sin\theta_i, \cos\theta_i]$ with $||\underline{x}_i||_2 = 1$, one easily finds that $\cos \widehat{X_i X_j} = \underline{x}_i^T \underline{x}_j$ and therefore

$$\widehat{X_i X_j} = \arccos[\tfrac{1}{2} \cos(\theta_i - \theta_j)\{1 + \cos(\phi_i - \phi_j)\} + \tfrac{1}{2} \cos(\theta_i + \theta_j)\{1 - \cos(\phi_i - \theta_j)\}] \quad (5.1)$$

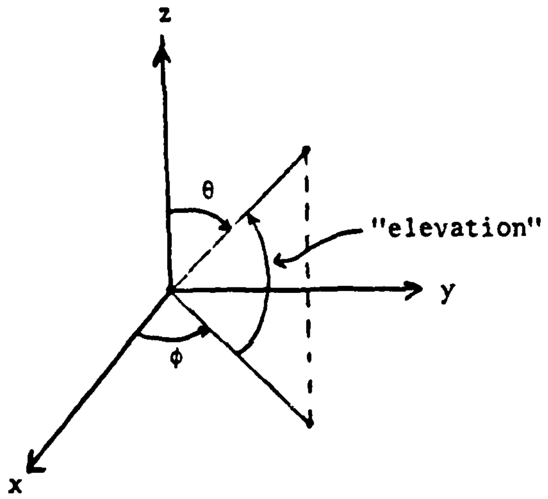

Figure 5.1

A summary table of the P-values using Table 6.1 is given below.

Cats	n	Y_n^{even}	P-value	Cats	n	Y_n^{even}	P-value
H_1	15	.4508	>> .20	$H_1 + H_2$	31	.9181	≈ .040
H_2	16	.7321	≈ .120				
V_1	16	.4923	>> .20				
				$V_1 + V_2$	32	.8603	≈ .056
V_2	16	.6239	> .20				
N_1	15	.6260	> .20				
				$N_1 + N_2$	31	.4421	>> .20
N_2	16	.6076	> .20				

Table 5.1

As the table shows, the test of uniformity is not significant for any individual cat. Grouping H_1 and H_2 or V_1 and V_2, however, does give significant results, whereas grouping N_1 and N_2 is still not significant.

§6. Graphs and Tables

In Table 6.1, the estimated smoothed percentage points for various values of n and significance levels α are presented, using the method described in Section 3. For $n = 2$ and $n = \infty$, the values are taken from the formulae in Theorem 2 and Table 6.2, respectively. For comparison purposes, we record the approximate significance points of Prentice (1978). The quantiles of the limiting distributions of Y^{odd}, Y^{even} and Y are given in Table 6.2, where $\alpha = P[Y > y]$, etc. More detailed values of the distributions are presented in Table 6.3, with step size 0.05. Figure 6.1 and Figure 6.2 show the graphs of the c.d.f. and the p.d.f. of Y^{odd}, Y^{even} and Y. All the values associated with the limiting distributions are calculated by the Laguerre transform method described in Section 4.

Table 6.1

Selected Smoothed Percentage Points for Y_n^{odd}, Y_n^{even} and Y_n

Sample Size	Statistics	.20	.10	.05	.025	.01	.001
2 (Exact)	Y_n^{odd}	1.4097	1.5903	1.7129	1.7978	1.8725	1.9597
	Y_n^{even}	0.6180	0.7225	0.8012	0.8585	0.9102	0.9715
	Y_n	1.9004	2.2084	2.4359	2.5991	2.7458	2.9195
5	Y_n^{odd}	1.4171	1.7788	2.0771	2.3456	2.6704	3.3946
	Y_n^{even}	0.6269	0.7345	0.8348	0.9323	1.0598	1.3838
	Y_n	1.9132	2.2767	2.6154	2.9387	3.3338	4.3277
10	Y_n^{odd}	1.4162	1.8063	2.1534	2.4802	2.8914	3.8606
	Y_n^{even}	0.6353	0.7490	0.8567	0.9619	1.1000	1.4476
	Y_n	1.9278	2.3124	2.6803	3.0389	3.4927	4.6366
20	Y_n^{odd}	1.4151	1.8135	2.1832	2.5385	2.9935	4.0913
	Y_n^{even}	0.6405	0.7582	0.8696	0.9777	1.1183	1.4657
	Y_n	1.9372	2.3327	2.7137	3.0866	3.5652	4.7610
40	Y_n^{odd}	1.4144	1.8155	2.1959	2.5654	3.0424	4.2060
	Y_n^{even}	0.6434	0.7634	0.8765	0.9858	1.1270	1.4712
	Y_n	1.9423	2.3434	2.7306	3.1098	3.5997	4.8156
100	Y_n^{odd}	1.4140	1.8161	2.2029	2.5808	3.0712	4.2747
	Y_n^{even}	0.6452	0.7666	0.8809	0.9908	1.1320	1.4735
	Y_n	1.9455	2.3501	2.7409	3.1236	3.6199	4.8460
∞	Y^{odd}	1.41363	1.81631	2.20727	2.59079	3.09000	4.32040
	Y^{even}	0.64643	0.76879	0.88384	0.99413	1.13534	1.47452
	Y	1.94776	2.35459	2.74772	3.13268	3.63309	4.86522
Prentice (1978)							
(÷4)	Y^{odd}	-	1.7750	2.1750	-	3.0500	4.3000
(÷16)	Y^{even}	-	0.7625	0.8813	-	1.1313	1.4680

Table 6.2. The Quantiles of the Limiting Distributions

α	Y(ODD)	Y(EVEN)	Y
0.999	0.18563	0.16005	0.47887
0.995	0.22343	0.18567	0.55003
0.990	0.24414	0.20184	0.58984
0.975	0.28309	0.22580	0.65944
0.950	0.32650	0.24992	0.73038
0.900	0.39123	0.28234	0.82748
0.850	0.44743	0.30846	0.90397
0.800	0.50073	0.33171	0.97212
0.750	0.55346	0.35351	1.03633
0.700	0.60686	0.37460	1.09897
0.650	0.66181	0.39553	1.16151
0.600	0.71904	0.41674	1.22509
0.550	0.77939	0.43850	1.29091
0.500	0.84376	0.46113	1.35992
0.450	0.91324	0.48501	1.43340
0.400	0.98921	0.51060	1.51289
0.350	1.07352	0.53849	1.60028
0.300	1.16916	0.56940	1.69861
0.250	1.28015	0.60472	1.81215
0.200	1.41363	0.64643	1.94776
0.150	1.58259	0.69823	2.11887
0.100	1.81631	0.76879	2.35459
0.050	2.20727	0.88384	2.74772
0.025	2.59079	0.99413	3.13268
0.010	3.09000	1.13534	3.63309
0.005	3.46337	1.23944	4.00709
0.001	4.32040	1.47452	4.86522

(e.g., $P[Y > 0.47887] = 0.999$)

Table 6.3. The Cumulative Distribution Functions of Y^{odd}, Y^{even} and Y

X	F-ODD(X)	F-EVEN(X)	F(X)	X	F-ODD(X)	F-EVEN(X)	F(X)
0.00	0.00000	0.00000	0.00000	2.55	0.97308	1.00000	0.92899
0.05	0.00000	0.00000	0.00000	2.60	0.97542	1.00000	0.93500
0.10	0.00000	0.00000	0.00000	2.65	0.97756	1.00000	0.94051
0.15	0.00005	0.00025	0.00000	2.70	0.97952	1.00000	0.94557
0.20	0.00180	0.00887	0.00000	2.75	0.98131	1.00000	0.95021
0.25	0.01147	0.05008	0.00000	2.80	0.98295	1.00000	0.95446
0.30	0.03353	0.13223	0.00000	2.85	0.98444	1.00000	0.95836
0.35	0.06670	0.24167	0.00000	2.90	0.98581	1.00000	0.96194
0.40	0.10739	0.36068	0.00006	2.95	0.98706	1.00000	0.96522
0.45	0.15235	0.47607	0.00040	3.00	0.98820	1.00000	0.96822
0.50	0.19930	0.58021	0.00170	3.05	0.98924	1.00000	0.97096
0.55	0.24673	0.66972	0.00500	3.10	0.99019	1.00000	0.97348
0.60	0.29367	0.74401	0.01152	3.15	0.99105	1.00000	0.97578
0.65	0.33945	0.80405	0.02228	3.20	0.99184	1.00000	0.97788
0.70	0.38368	0.85158	0.03792	3.25	0.99257	1.00000	0.97981
0.75	0.42606	0.88860	0.05860	3.30	0.99323	1.00000	0.98157
0.80	0.46643	0.91703	0.08408	3.35	0.99383	1.00000	0.98318
0.85	0.50471	0.93863	0.11385	3.40	0.99438	1.00000	0.98465
0.90	0.54085	0.95487	0.14718	3.45	0.99488	1.00000	0.98600
0.95	0.57486	0.96699	0.18331	3.50	0.99533	1.00000	0.98723
1.00	0.60678	0.97597	0.22144	3.55	0.99575	1.00000	0.98835
1.05	0.63665	0.98258	0.26083	3.60	0.99613	1.00000	0.98937
1.10	0.66455	0.98743	0.30083	3.65	0.99647	1.00000	0.99031
1.15	0.69056	0.99095	0.34085	3.70	0.99679	1.00000	0.99116
1.20	0.71476	0.99351	0.38043	3.75	0.99707	1.00000	0.99194
1.25	0.73726	0.99536	0.41917	3.80	0.99734	1.00000	0.99264
1.30	0.75813	0.99669	0.45670	3.85	0.99757	1.00000	0.99331
1.35	0.77748	0.99764	0.49303	3.90	0.99779	1.00000	0.99390
1.40	0.79539	0.99833	0.52776	3.95	0.99799	1.00000	0.99444
1.45	0.81196	0.99882	0.56085	4.00	0.99817	1.00000	0.99493
1.50	0.82727	0.99916	0.59224	4.05	0.99833	1.00000	0.99538
1.55	0.84141	0.99941	0.62191	4.10	0.99848	1.00000	0.99580
1.60	0.85445	0.99958	0.64985	4.15	0.99862	1.00000	0.99617
1.65	0.86647	0.99971	0.67609	4.20	0.99874	1.00000	0.99651
1.70	0.87755	0.99979	0.70067	4.25	0.99886	1.00000	0.99682
1.75	0.88774	0.99986	0.72364	4.30	0.99896	1.00000	0.99711
1.80	0.89713	0.99990	0.74506	4.35	0.99905	1.00000	0.99737
1.85	0.90576	0.99993	0.76501	4.40	0.99914	1.00000	0.99760
1.90	0.91369	0.99995	0.78355	4.45	0.99922	1.00000	0.99782
1.95	0.92097	0.99997	0.80076	4.50	0.99929	1.00000	0.99801
2.00	0.92767	0.99998	0.81671	4.55	0.99935	1.00000	0.99819
2.05	0.93381	0.99998	0.83148	4.60	0.99941	1.00000	0.99835
2.10	0.93944	0.99999	0.84514	4.65	0.99946	1.00000	0.99850
2.15	0.94461	0.99999	0.85777	4.70	0.99951	1.00000	0.99863
2.20	0.94935	0.99999	0.86942	4.75	0.99956	1.00000	0.99876
2.25	0.95370	1.00000	0.88017	4.80	0.99960	1.00000	0.99887
2.30	0.95768	1.00000	0.89008	4.85	0.99963	1.00000	0.99897
2.35	0.96133	1.00000	0.89921	4.90	0.99967	1.00000	0.99906
2.40	0.96467	1.00000	0.90762	4.95	0.99970	1.00000	0.99915
2.45	0.96772	1.00000	0.91535	5.00	0.99972	1.00000	0.99923
2.50	0.97052	1.00000	0.92246				

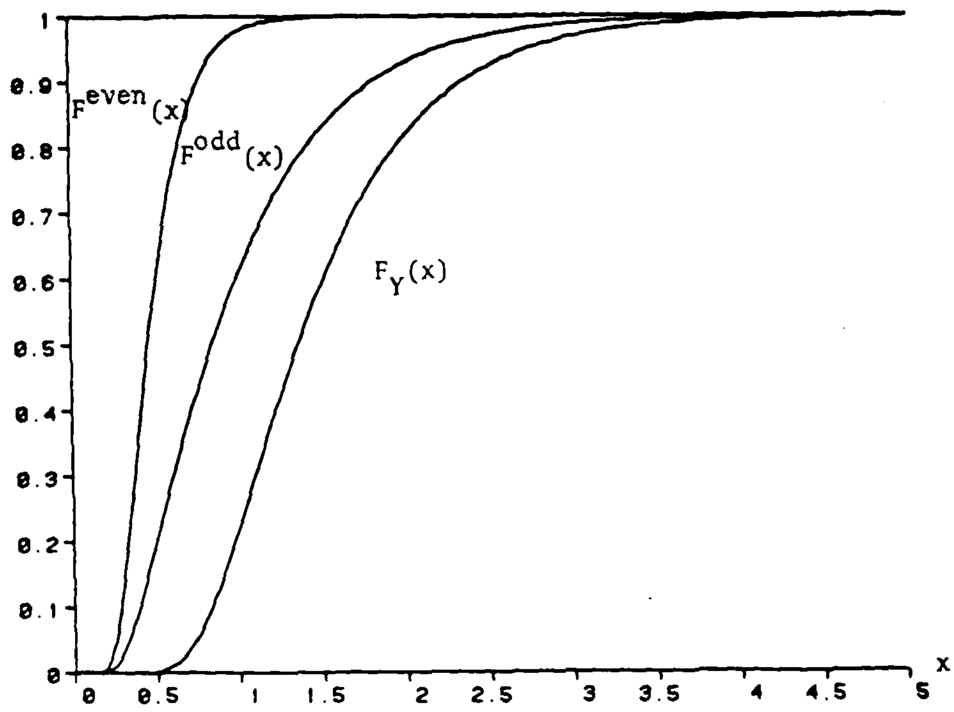

Fig. 6.1. The cumulative distribution functions of Y^{odd}, Y^{even} and Y

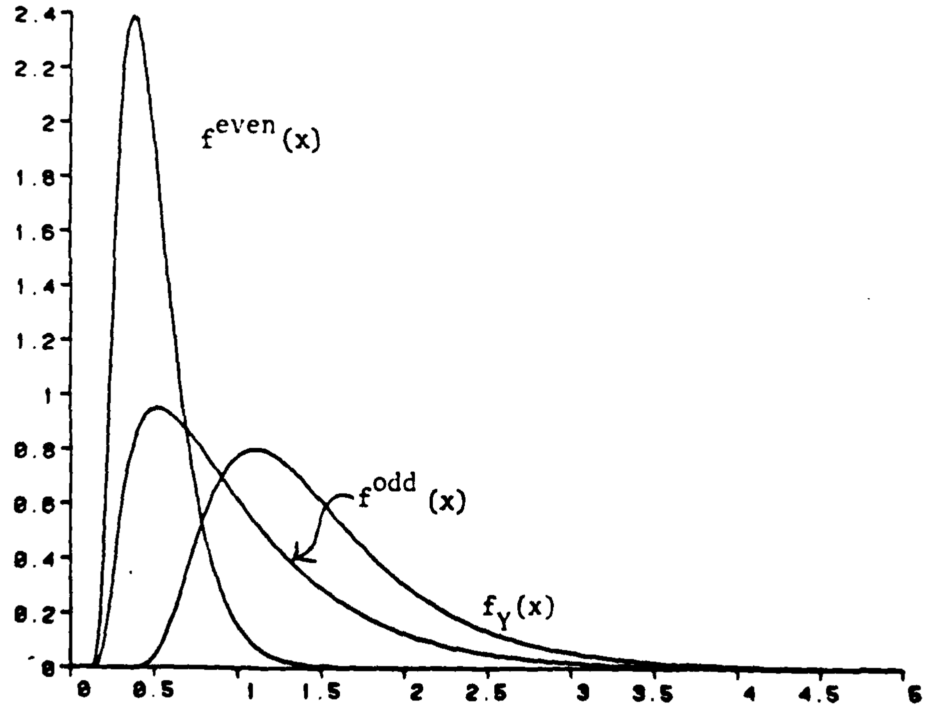

Fig. 6.2. The probability density functions of Y^{odd}, Y^{even} and Y

APPENDIX A

The Laguerre Transform

The Laguerre polynomials $L_n(x)$, defined by the Rodrigues formula $L_n(x) = \frac{1}{n!} e^x (\frac{d}{dx})^n (x^n e^{-x})$, form a set of orthonormal polynomials with weighting function $w(x) = e^{-x}$ on $(0, \infty)$ (see, e.g., Szego (1975)). The associated Laguerre functions $\ell_n(x) = e^{-\frac{1}{2}x} L_n(x)$ then provide an orthonormal basis in $L_2(0, \infty)$. For any $f(x) \in L_2(0, \infty)$, one has the Fourier-Laguerre expansion

$$\text{(A.1)} \qquad f(x) = \sum_{n=0}^{\infty} f_n^{\dagger} \ell_n(x) \ ; \quad f_n^{\dagger} = \int_0^{\infty} f(x) \ell_n(x) dx \ .$$

Let $T_f^{\dagger}(u) = \sum_{n=0}^{\infty} f_n^{\dagger} u^n$ and $T_f^{\#}(u) = \sum_{n=0}^{\infty} f_n^{\#} u^n \overset{\text{def}}{\equiv} (1-u) T_f^{\dagger}(u)$. Since $\sum_{n=0}^{\infty} \ell_n(x) u^n = (1-u)^{-1} \exp\{-\frac{1}{2}x(1+u)(1-u)^{-1}\}$, one has, when $f(x)$ is integrable on $(0, \infty)$,

$$T_f^{\#}(u) = (1-u) \sum_{n=0}^{\infty} f_n^{\dagger} u^n = (1-u) \sum_{n=0}^{\infty} u^n \int_0^{\infty} f(x) \ell_n(x) dx$$

$$= \int_0^{\infty} f(x) \exp\{-\tfrac{1}{2}x(1+u)(1-u)^{-1}\} dx \ , \quad 0 \le u < 1 \ ,$$

i.e.,

$$\text{(A.2)} \qquad T_f^{\#}(u) = \phi\left(\frac{1+u}{2(1-u)}\right)$$

where $\phi(s) = \int_0^{\infty} e^{-sx} f(x) dx$ is the Laplace transform of $f(x)$. Let $f(x)*g(x) = \int_0^{x} f(x-y) g(y) dy$. Since $\phi_{f*g}(s) = \phi_f(x) \phi_g(s)$,

$$T^{\#}_{f*g}(u) = T^{\#}_{f}(u) T^{\#}_{g}(u) \tag{A.3}$$

for functions f(x), g(x) that are both in $L_1(0, \infty)$ and $L_2(0, \infty)$. From (A.3) one obtains

$$(f*g)^{\#}_{n} = \sum_{m=0}^{n} f^{\#}_{n-m} g^{\#}_{m} \quad . \tag{A.4}$$

The transformation via (A.3) maps functions f(x), g(x) into sequences $(f^{\#}_{n})(g^{\#}_{n})$, and their continuum convolution f(x)*g(x) is mapped into a lattice convolution and then back onto the continuum via $f^{\dagger}_{n} = \sum_{m=0}^{n} f^{\#}_{m}$ and the representation (A.1). This transformation procedure was introduced originally in Keilson and Nunn (1980). The Laguerre transform was extended subsequently in Keilson, Nunn and Sumita (1981) to handle functions on the full continuum, and further studied by Sumita (1981).

APPENDIX B

Asymptotic Behavior of $\bar{F}_Y(x) = P[Y > x]$ as $x \to +\infty$

In this appendix, we derive the asymptotic expression of $\bar{F}_Y(x)$ as $x \to +\infty$. The Laplace transform of the p.d.f. of Y is given by

$$\text{(B.1)} \qquad \phi_Y(s) = \prod_{j=1}^{\infty} \left(\frac{\theta_j}{\theta_j + s}\right)^{j+\frac{1}{2}} ; \quad \theta_j = \frac{1}{2a_j^2} .$$

Let

$$\text{(B.2)} \qquad \alpha(s) = \left(\frac{\theta_1}{\theta_1 + s}\right)^{3/2} ; \quad \lambda(s) = \prod_{j=2}^{\infty} \left(\frac{\theta_j}{\theta_j + s}\right)^{j+\frac{1}{2}} .$$

We note that $\lambda(s)$ is regular for $Re(s) > -\theta_2$. Let $\bar{F}_Y(x) = P[Y > x]$ and define $\psi(w) = L\{e^{\theta_1 x}\bar{F}_Y(x)\} = \int_0^{\infty} e^{-(w - \theta_1)x}\bar{F}_Y(x)dx$. Then, from (B.1) and (B.2), one has

$$\text{(B.3)} \qquad \Psi(w) = \frac{1 - \phi_Y(w - \theta_1)}{w - \theta_1} = \frac{1 - \alpha(w - \theta_1)}{w - \theta_1} \cdot \lambda(w - \theta_1) + \frac{1 - \lambda(w - \theta_1)}{w - \theta_1} .$$

After a little algebra, one finds, from (B.2), that

$$\text{(B.4)} \qquad \frac{1 - \alpha(w - \theta_1)}{w - \theta_1} = \frac{\sqrt{\theta_1}}{w^{3/2}} + \frac{1}{\sqrt{\theta_1 w}} + \sum_{n=3}^{\infty} (-1)^n \left(\frac{1}{\theta}\right)^{\frac{n-1}{2}} w^{\frac{n-3}{2}} .$$

On the other hand, $\lambda(w - \theta_1)$ and $\frac{1 - \lambda(w - \theta_1)}{w - \theta_1}$ are regular for $Re(w) > \theta_1 - \theta_2$ with $\theta_1 - \theta_2 < 0$. Hence, one has the Taylor expansion

$$\text{(B.5)} \qquad \lambda(w - \theta_1) = \sum_{n=0}^{\infty} \lambda_n w^n .$$

From (B.3), (B.4), (B.5) and the regularity of $\frac{1 - \lambda(w - \theta_1)}{w - \theta_1}$ for Re(w) > $\theta_1 - \theta_2$, we obtain the asymptotic expression of $\psi(w)$ near zero, i.e.,

$$\text{(B.6)} \qquad \psi(w) \sim \frac{\lambda_0\sqrt{\theta_1}}{w^{3/2}} + \left(\frac{\lambda_0}{\sqrt{\theta_1}} + \lambda_1\sqrt{\theta_1}\right)\frac{1}{\sqrt{w}} \quad \text{as} \quad w \to 0+ \quad .$$

This then implies that (see, e.g., Widder (1946), p. 192)

$$^{\dagger}\text{(B.7)} \qquad \bar{F}_Y(x) \sim \frac{e^{-2x}}{\sqrt{\pi}}\left[2\lambda_0\sqrt{2x} + \left(\frac{\lambda_0}{\sqrt{2}} + \lambda_1\sqrt{2}\right)\frac{1}{\sqrt{x}}\right] \quad \text{as} \quad x \to +\infty \quad ,$$

where $\theta_1 = 2$ is substituted. The constants λ_0 and λ_1 are obtained from $\lambda(w - \theta_1) = \prod_{j=2}^{\infty} \left(\frac{\theta_j}{\theta_j - \theta_1 + w}\right)^{j+\frac{1}{2}} = \sum_{n=0}^{\infty} \lambda_n w^n$, i.e.,

$$\text{(B.8)} \qquad \begin{cases} \lambda_0 = \prod_{j=2}^{\infty} \left(\frac{\theta_j}{\theta_j - \theta_1}\right)^{j+\frac{1}{2}} \approx 4.952213394 \\ \\ \lambda_1 = -\sum_{j=2}^{\infty} \frac{j+\frac{1}{2}}{\theta_j - \theta_1} \prod_{\substack{r=2 \\ r\neq j}}^{\infty} \left(\frac{\theta_r}{\theta_r - \theta_1}\right)^{r+\frac{1}{2}} \approx -4.271400146 \end{cases} .$$

† The asymptotic expression in (B.7) agrees with Zolotarev (1961).

APPENDIX C

Data of Coleman's Cats Experiment

H_1		H_2		V_1		V_2		N_1		N_2	
elevation	ϕ	elevation	ϕ	elevation	ϕ	elevation	ϕ	elevation	ϕ	elevation	ϕ
0	41	17	41	9	3	72	6	6	85	-51	13
-25	-13	- 1	4	-30	-86	30	79	-4	33	-23	-73
54	- 4	-19	-16	-19	72	8	6	12	- 1	-18	-20
-58	47	-53	-57	14	-28	5	-27	31	-81	38	59
-18	-65	-28	-10	41	43	-11	24	-36	-76	-68	-73
27	54	-47	- 2	52	77	23	-66	30	-48	68	30
-10	12	-18	-54	-33	2	-41	28	5	31	27	-59
-20	15	16	-24	-20	- 5	-35	62	-24	-80	-66	-87
- 8	26	-17	23	- 5	16	- 2	69	44	84	-65	-37
-60	84	-73	14	-27	3	- 7	20	-13	-10	-57	- 1
9	51	17	49	2	71	30	44	- 2	55	24	57
21	-26	10	67	54	-75	- 8	37	49	38	-35	-40
-40	-23	22	68	-33	65	-51	14	22	64	-24	-12
- 5	-70	-36	-56	-26	70	-62	-18	39	-81	-28	-14
- 5	-54	3	-43	28	-24	12	49	16	-44	9	10
		- 4	-68	46	53	7	88			16	45

Acknowledgment

The authors are indebted to Professor Paul Coleman of the University of Rochester for motivating this study and for providing the data in Appendix C. The authors also wish to thank L. Ziegenfuss for her editorial contribution. A referee's comments and suggestions are gratefully acknowledged.

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